Arithmetic and Geometry of Markov Polynomials

Sam Evans
joint work with A.P. Veselov and B. Winn

Loughborough University

25th October, 2024

Markov equation

Markov Diophantine equation

$$X^2 + Y^2 + Z^2 = 3XYZ, \quad X, Y, Z \in \mathbb{Z}_+.$$

Markov Diophantine equation

$$X^2 + Y^2 + Z^2 = 3XYZ, \quad X, Y, Z \in \mathbb{Z}_+.$$

Markov 1880: Every solution can be found from $(1, 1, 1)$ by applying Vieta involution

$$(X, Y, Z) \to \left(X, Y, \frac{X^2 + Y^2}{Z} \right)$$

and permutations.

Generalised Markov equation (**Propp et al. 2003**)

$$X^2 + Y^2 + Z^2 = k(x, y, z)XYZ, \qquad k(x, y, z) = \frac{x^2 + y^2 + z^2}{xyz}$$

Generalised Markov equation (**Propp et al. 2003**)

$$X^2 + Y^2 + Z^2 = k(x, y, z)XYZ, \qquad k(x, y, z) = \frac{x^2 + y^2 + z^2}{xyz}$$

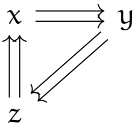
Using the same procedure applied to $(X = x, Y = y, Z = z)$, we get the solutions, which are Laurent polynomials of the parameters x, y, z. Indeed, we can use the alternative Vieta involution

$$(X, Y, Z) \to (X, Y, k(x, y, z)XY - Z).$$

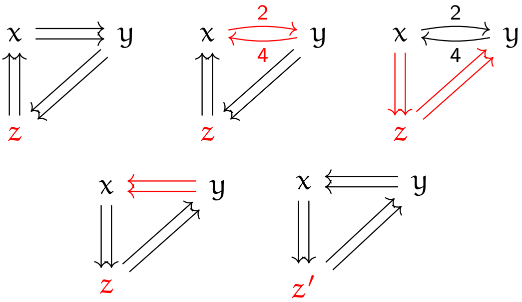
These Laurent polynomials are called **Markov polynomials**.

Markov Cluster Algebra

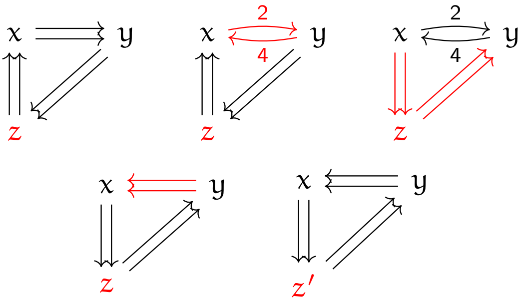
Markov quiver:



Markov quiver mutations:



Markov quiver mutations:



Seed mutation exchange relation:

$$z' = \frac{1}{z}\left(\prod_{x_i \to z} x_i + \prod_{z \to x_j} x_j\right)$$
$$= \frac{x^2 + y^2}{z}$$

Frobenius 1913: The Markov numbers can be indexed by the rationals in $[0, 1]$.

$$\rho = \frac{a}{b} \to m_\rho \quad \text{(Markov number)}$$

Frobenius 1913: The Markov numbers can be indexed by the rationals in $[0, 1]$.

$$\rho = \frac{a}{b} \to m_\rho \quad \text{(Markov number)}$$

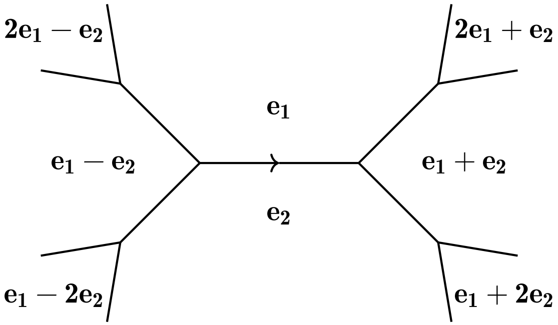


Figure: Conway Topograph

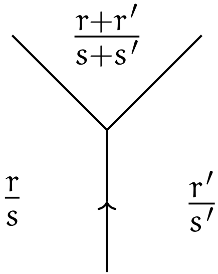


Figure: Farey rationals iterations on the Conway topograph

Conway Topograph and Frobenius Correspondence

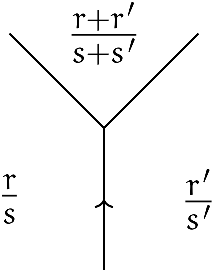


Figure: Farey rationals iterations on the Conway topograph

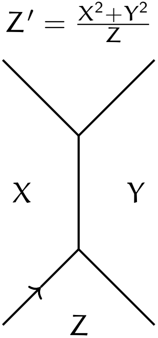
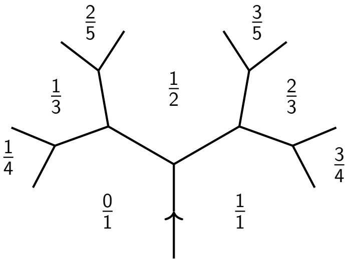
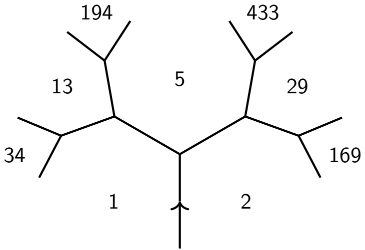
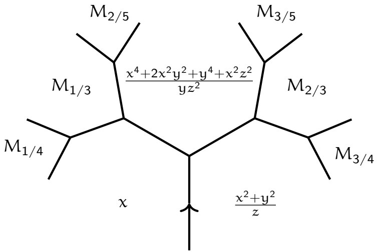


Figure: Markov number iterations on the Conway topograh

Conway Topograph and Frobenius Correspondence







Geometry of Markov Polynomials

$$M_\rho(x, y, z) = \frac{P_\rho(x, y, z)}{Q_\rho(x, y, z)}$$

Geometry of Markov Polynomials

$$M_\rho(x,y,z) = \frac{P_\rho(x,y,z)}{Q_\rho(x,y,z)}$$

Theorem 1 (EVW 2024)

The denominator of a Markov polynomial corresponding to the rational $\rho = \frac{a}{b}$ *is* $Q_\rho(x,y,z) = x^{(a-1)}y^{(b-1)}z^{(a+b-1)}$.

$$M_\rho(x, y, z) = \frac{P_\rho(x, y, z)}{Q_\rho(x, y, z)}$$

Theorem 1 (EVW 2024)

The denominator of a Markov polynomial corresponding to the rational $\rho = \frac{a}{b}$ *is* $Q_\rho(x, y, z) = x^{(a-1)} y^{(b-1)} z^{(a+b-1)}$.

By homogeneity we have

$$P_\rho(x, y, z) = \sum A_{ij} x^{2i} y^{2j} z^{2(a+b-1-i-j)}.$$

$$M_\rho(x, y, z) = \frac{P_\rho(x, y, z)}{Q_\rho(x, y, z)}$$

Theorem 1 (EVW 2024)

The denominator of a Markov polynomial corresponding to the rational $\rho = \frac{a}{b}$ *is* $Q_\rho(x, y, z) = x^{(a-1)} y^{(b-1)} z^{(a+b-1)}$.

By homogeneity we have

$$P_\rho(x, y, z) = \sum A_{ij} x^{2i} y^{2j} z^{2(a+b-1-i-j)}.$$

Propp 2005: Markov polynomials have positive coefficients.

Geometry of Markov Polynomials

$$M_\rho(x,y,z) = \frac{P_\rho(x,y,z)}{Q_\rho(x,y,z)}$$

Theorem 1 (EVW 2024)

The denominator of a Markov polynomial corresponding to the rational $\rho = \frac{a}{b}$ *is* $Q_\rho(x,y,z) = x^{(a-1)}y^{(b-1)}z^{(a+b-1)}$.

By homogeneity we have

$$P_\rho(x,y,z) = \sum A_{ij}x^{2i}y^{2j}z^{2(a+b-1-i-j)}.$$

Propp 2005: Markov polynomials have positive coefficients.

We define the Newton polygon Δ_ρ as follows

$$\Delta_\rho = \Delta(M_\rho) := Conv\{(i,j) : A_{ij} \neq 0\} \subset \mathbb{Z}^2.$$

Newton Polygon Example

Example:

$$\rho = \frac{2}{3}, \;\; m_\rho = 29.$$

$$\begin{aligned} P_\rho(x, y, 1) = \\ x^8 + 4x^6y^2 + 6x^4y^4 + 4x^2y^6 \\ + y^8 + 2x^6 + 5x^4y^2 \\ + 4x^2y^4 + y^6 + x^4 \end{aligned}$$

Newton Polygon Example

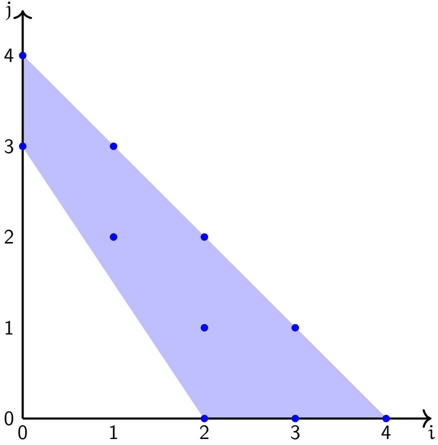


Figure: Newton polygon Δ_ρ.

Example:

$$\rho = \frac{2}{3}, \;\; m_\rho = 29.$$

$$P_\rho(x, y, 1) = \\ x^8 + 4x^6y^2 + 6x^4y^4 + 4x^2y^6 \\ + y^8 + 2x^6 + 5x^4y^2 \\ + 4x^2y^4 + y^6 + x^4$$

Theorem 2 (EVW 2024)

Given a rational $\rho = \frac{a}{b}$, Δ_ρ *is the area (on the* ij*-plane with* $i, j \geqslant 0$*) satisfying the conditions*

$$\Delta_\rho = \begin{cases} \dfrac{i}{a} + \dfrac{j}{b} \geqslant 1 \\ i + j \leqslant a + b - 1 \end{cases}$$

Geometry of Newton Polygon

Theorem 2 (EVW 2024)

Given a rational $\rho = \frac{a}{b}$, Δ_ρ *is the area (on the* ij*-plane with* $i, j \geqslant 0$*) satisfying the conditions*

$$\Delta_\rho = \begin{cases} \dfrac{i}{a} + \dfrac{j}{b} \geqslant 1 \\ i + j \leqslant a + b - 1 \end{cases}$$

Conjecture 3 (Saturation Conjecture, EVW 2024)

Terms that appear in the numerator of a Markov polynomial M_ρ *are precisely those corresponding to the set of integer lattice points on* Δ_ρ.

Weighted Newton Polygon

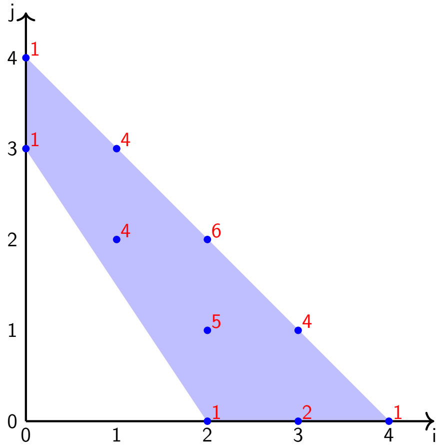


Figure: 'Weighted' Newton polygon $\Delta_\rho, \rho = \frac{2}{3}$.

$$P_\rho(x, y, 1) = x^8 + 4x^6y^2 + 6x^4y^4 + 4x^2y^6 + y^8 + 2x^6 + 5x^4y^2 + 4x^2y^4 + y^6 + x^4$$

Weighted Newton Polygon

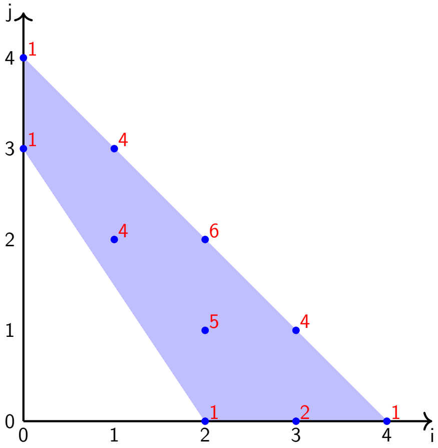


Figure: 'Weighted' Newton polygon $\Delta_\rho, \rho = \frac{2}{3}$.

$$
\begin{aligned}
P_\rho(x, y, 1) = \\
x^8 + 4x^6y^2 + 6x^4y^4 + 4x^2y^6 \\
+ y^8 + 2x^6 + 5x^4y^2 \\
+ 4x^2y^4 + y^6 + x^4
\end{aligned}
$$

We define the **Markov function** on the Newton polygon

$$
\begin{aligned}
\mathcal{M} : \Delta_\rho &\to \mathbb{Z} \\
(i, j) &\mapsto \mathcal{M}((i, j)).
\end{aligned}
$$

Coefficients on Newton Polygon Boundary

Theorem 4 (EVW 2024)

Given a rational $\frac{a}{b}$ the coefficients on the boundary of the corresponding Markov polynomial's Newton polygon are binomial coefficients. In particular,

Line	*Coefficients*
$j = 0$	$\binom{b-1}{i-a}$
$i = 0$	$\binom{a-1}{j-b}$
$i + j = a + b - 1$	$\binom{a+b-1}{i}$

Coefficients on Newton Polygon Interior

Coefficients of second upper-most diagonal of Δ_ρ

$$[2, 5, 4, 1] = [1, 3, 3, 1] + [1, 2, 1, 0]$$

Coefficients on Newton Polygon Interior

Coefficients of second upper-most diagonal of Δ_ρ

$$[2, 5, 4, 1] = [1, 3, 3, 1] + [1, 2, 1, 0]$$

Theorem 5 (EVW 2024)

Coefficients on the 2nd upper-most diagonal:

$$(a-1)\binom{a+b-2}{i} + (b-a)\binom{a+b-3}{i-1}.$$

Coefficients on Newton Polygon Interior

Coefficients of second upper-most diagonal of Δ_ρ

$$[2, 5, 4, 1] = [1, 3, 3, 1] + [1, 2, 1, 0]$$

Theorem 5 (EVW 2024)

Coefficients on the 2nd upper-most diagonal:

$$(a-1)\binom{a+b-2}{i} + (b-a)\binom{a+b-3}{i-1}.$$

Theorem 6 (EVW 2024)

Coefficients on the 3rd upper-most diagonal:

$$\frac{(a-1)(a-2)}{2}\binom{a+b-3}{i} + [a(b-a)-a]\binom{a+b-4}{i-1} + \frac{1}{2}[(b-a)^2+5a-3b]\binom{a+b-5}{i-2}.$$

Coefficients on Newton Polygon Interior

Theorem 7 (EVW 2024)

Coefficients on the 2nd lower-most horizontal of the Newton polygon of Markov polynomials (the line $j = 1$*) are*

$$(3a-1)\binom{b-2}{i-a} + (b-2a)\binom{b-3}{i-1-a}.$$

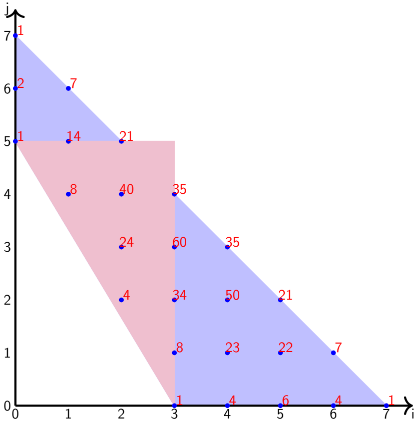


Figure: ‘Weighted’ Newton polygon $\Delta_\rho, \rho = \frac{3}{5}(m_\rho = 433)$.

Coefficients on Critical Triangle

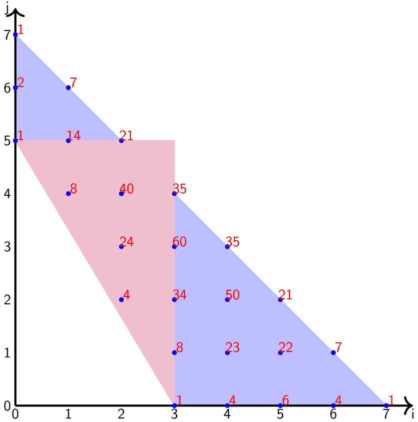


Figure: 'Weighted' Newton polygon $\Delta_\rho, \rho = \frac{3}{5}(m_\rho = 433)$.

Conjecture 8 (EVW 2024)

The coefficients of the Markov polynomial $M_\rho, \rho = \frac{a}{b}$ lying inside the critical triangle of the Newton polygon are all multiples of 4.

Fibonacci Polynomials

Markov polynomials M_ρ, with $\rho = \frac{1}{n+1}$, are a specialisation of the Fibonacci polynomials previously studied by **Caldero, Zelevinsky (2006)**.

Fibonacci Polynomials

Markov polynomials M_ρ, with $\rho = \frac{1}{n+1}$, are a specialisation of the Fibonacci polynomials previously studied by **Caldero, Zelevinsky (2006)**.

Corollary 9

The Markov polynomials $M_\rho, \rho = \frac{1}{n+1}$ *have coefficients*

$$A_{ij} = \binom{n-j}{n+1-i-j}\binom{i+j}{j}.$$

Pell Polynomials

The next 'simplest' case of Markov polynomials would be those corresponding to Pell numbers, namely M_ρ for $\rho = \frac{n}{n+1}$.

The next ‘simplest’ case of Markov polynomials would be those corresponding to Pell numbers, namely M_ρ for $\rho = \frac{n}{n+1}$.

It is known that triples of the form

$$(2, P_{2k-1}, P_{2k+1})$$

are Markov triples, where P_i's are Pell numbers.

Pell Polynomials

The next 'simplest' case of Markov polynomials would be those corresponding to Pell numbers, namely M_ρ for $\rho = \frac{n}{n+1}$.

It is known that triples of the form

$$(2, P_{2k-1}, P_{2k+1})$$

are Markov triples, where P_i's are Pell numbers.

The corresponding Markov polynomial triple has the form

$$(M_{1/1}, M_{k-1/k}, M_{k/k+1}).$$

Pell Polynomials

Applying the Vieta involution inductively, one obtains the following recursive formulas:

$$\begin{aligned} R_{2k+1} &= (x^2+y^2)R_{2k} + y^2z^2R_{2k-1} \\ R_{2k} &= (x^2+y^2)R_{2k-1} + x^2z^2R_{2k-2}, \end{aligned}$$

with $R_1 = 1, R_3 = x^4 + 2x^2y^2 + y^4 + x^2z^2$, where R_{2k+1} denotes the numerator of the Markov polynomial $M_{k/k+1}$.

Applying the Vieta involution inductively, one obtains the following recursive formulas:

$$R_{2k+1} = (x^2 + y^2)R_{2k} + y^2z^2R_{2k-1}$$
$$R_{2k} = (x^2 + y^2)R_{2k-1} + x^2z^2R_{2k-2},$$

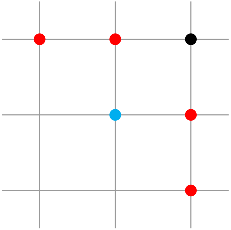
with $R_1 = 1, R_3 = x^4 + 2x^2y^2 + y^4 + x^2z^2$, where R_{2k+1} denotes the numerator of the Markov polynomial $M_{k/k+1}$.

From this we can produce a recursive method for calculating specific coefficients

$$A_{i,j}^{(2k+1)} = \left[A_{i-2,j}^{(2k-1)} + 2A_{i-1,j-1}^{(2k-1)} + A_{i,j-2}^{(2k-1)}\right] + \left[A_{i-1,j}^{(2k-1)} + A_{i,j-1}^{(2k-1)}\right] - A_{i-1,j-1}^{(2k-3)}.$$

Pell Polynomials

$$A_{i,j}^{(2k+1)} = \left[A_{i-2,j}^{(2k-1)} + 2A_{i-1,j-1}^{(2k-1)} + A_{i,j-2}^{(2k-1)}\right] + \left[A_{i-1,j}^{(2k-1)} + A_{i,j-1}^{(2k-1)}\right] - A_{i-1,j-1}^{(2k-3)}.$$



Klein Diagram for Continued Fractions

Consider $\rho = \frac{5}{3} = [1, 1, 2]$. Table

of convergents:

			1	1	2
p_k	0	1	1	2	5
q_k	1	0	1	1	3

Klein Diagram for Continued Fractions

Consider $\rho = \frac{5}{3} = [1, 1, 2]$. Table

of convergents:

			1	1	2
p_k	0	1	1	2	5
q_k	1	0	1	1	3

We have sails $A_0A_1A_2\dots$ and $B_0B_1B_2\dots$

$$A_i = (q_{2i-1}, p_{2i-1}),$$
$$B_i = (q_{2i}, p_{2i}).$$

In our example,

$A_0 = (1, 0), A_1 = (1, 1), A_2 = (5, 3)$
$B_0 = (0, 1), B_1 = (2, 1), [B_2 = (5, 3)]$

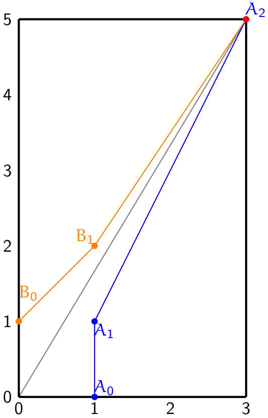
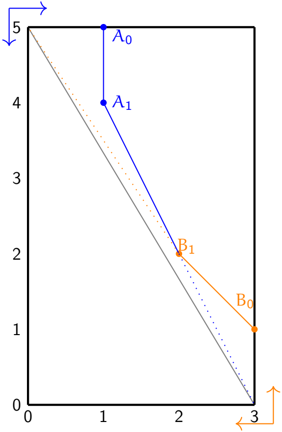


Figure: Klein Diagram for $\rho = \frac{5}{3}$

Karpenkov 2013: We have the following Edge-Angle Duality

$$\mathrm{l}\alpha(\angle A_i A_{i+1} A_{i+2}) = \mathrm{l}\ell(B_i B_{i+1}) \qquad (= a_{2i+2}),$$
$$\mathrm{l}\alpha(\angle B_i B_{i+1} B_{i+2}) = \mathrm{l}\ell(A_{i+1} A_{i+2}) \qquad (= a_{2i+3}),$$

Markov Sails



$$A_i := (q_{2i-1}, b - p_{2i-1}), \qquad B_i := (a - q_{2i}, p_{2i}),$$

Coefficients on the Markov Sail

Conjecture 10 (EVW 2024)

Coefficients on the edge $C_i C_{i+1}$ *of the Markov sail are arithmetic progressions with differences* $d(C_i C_{i+1})$ *satisfying*

$$d(B_i B_{i+1}) = -\mathcal{M}(A_{i+1}), \qquad d(A_{i+1} A_{i+2}) = -\mathcal{M}(B_{i+1}).$$

Coefficients on the Markov Sail

Conjecture 10 (EVW 2024)

Coefficients on the edge $C_i C_{i+1}$ of the Markov sail are arithmetic progressions with differences $d(C_i C_{i+1})$ satisfying

$$d(B_i B_{i+1}) = -\mathcal{M}(A_{i+1}), \qquad d(A_{i+1} A_{i+2}) = -\mathcal{M}(B_{i+1}).$$

Conjecture 11 (EVW 2024)

Consider the continued fraction $\frac{b}{a} = [a_1, a_2, \ldots, a_n]$. If $n = 2m+1$ (odd) then $\mathcal{M}(B_m) = 4$. If $n = 2m$ (even) then $\mathcal{M}(A_m) = 4$.

Both of these conjectures are proven in the case of the Pell polynomials. Combining these we obtain

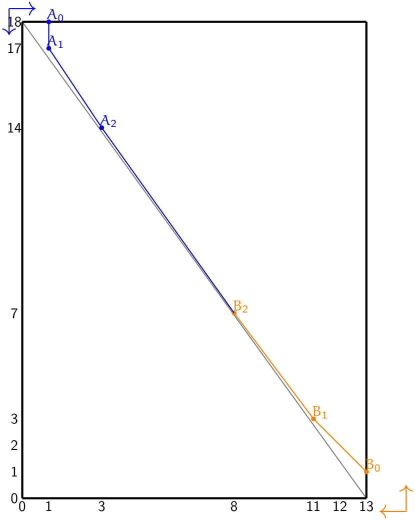
Theorem 12

The coefficients on the Markov sail corresponding to a rational of the form $\frac{n}{n+1}$ *are (from top to bottom)*

$$(7n-10,\ 4,\ 8,\ \ldots,\ 4n-4,\ 3n-1).$$

Markov Sail Example

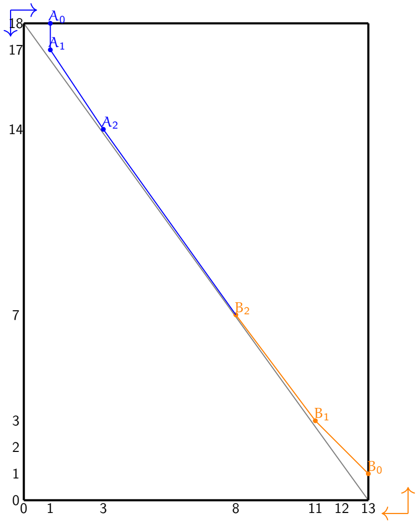
$$\frac{b}{a} = \frac{18}{13} = [1, 2, 1, 1, 2]$$



Markov Sail Example

$$\frac{b}{a} = \frac{18}{13} = [1, 2, 1, 1, 2]$$

Conjecture 11 $\implies \mathcal{M}(B_2) = 4.$



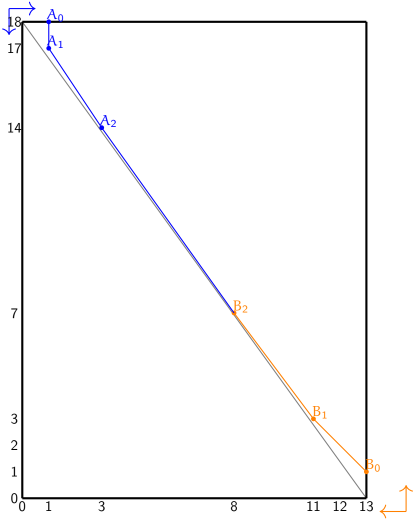
Markov Sail Example

$$\frac{b}{a} = \frac{18}{13} = [1, 2, 1, 1, 2]$$

Conjecture 11 $\implies \mathcal{M}(B_2) = 4.$

Now applying Conjecture 10 recursively,

$$\mathcal{M}(A_2) = \mathcal{M}(B_2) + (a_5 - 1)\mathcal{M}(B_2) = 8$$
$$\mathcal{M}(B_1) = \mathcal{M}(B_2) + a_4\mathcal{M}(A_2) = 12$$
$$\mathcal{M}(A_1) = \mathcal{M}(A_2) + a_3\mathcal{M}(B_1) = 20.$$



Log-Concavity of Coefficients

A sequence $x = (x_0, x_1, \dots, x_n)$ is said to be **log-concave** if it satisfies the property

$$x_k^2 \geqslant x_{k-1}x_{k+1},$$

for $k \in \{1, 2, \dots, n-1\}$.

Log-Concavity of Coefficients

A sequence $x = (x_0, x_1, \ldots, x_n)$ is said to be **log-concave** if it satisfies the property

$$x_k^2 \geqslant x_{k-1}x_{k+1},$$

for $k \in \{1, 2, \ldots, n-1\}$.

Theorem 13 (EVW 2024)

The sequence of coefficients that appear on the 2nd upper diagonal of the Newton polygon associated to a Markov polynomial is (strictly) log-concave.

We say that a weighted lattice is **weakly log-concave** if the sequence of weights in all principal directions are log-concave.

Log-Concavity of Coefficients

We say that a weighted lattice is **weakly log-concave** if the sequence of weights in all principal directions are log-concave.

Conjecture 14 (EVW 2024)

Coefficients of Markov polynomials are weakly log-concave.

Log-Concavity of Coefficients

We say that a weighted lattice is **weakly log-concave** if the sequence of weights in all principal directions are log-concave.

Conjecture 14 (EVW 2024)

Coefficients of Markov polynomials are weakly log-concave.

Theorem 15 (EVW 2024)

The above holds in the case $\rho = \frac{1}{n+1}$.

Combinatorial Interpretation of Markov

Markov numbers can be interpreted combinatorially, as perfect matching on snake graphs. To construct Markov numbers in this way we first look at the corresponding rational on the traingular lattice.

Markov numbers can be interpreted combinatorially, as perfect matching on snake graphs. To construct Markov numbers in this way we first look at the corresponding rational on the traingular lattice.

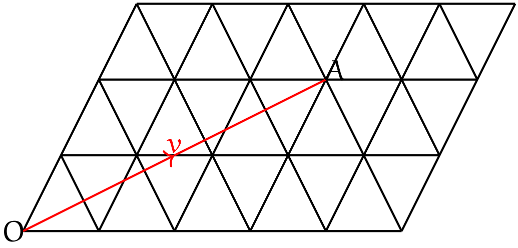


Figure: Triangular lattice, with a primitive vector v corresponding to the rational $\frac{2}{3}$ shown.

Take the intersection of the triangles that the primitive vector passes through to find the *Markov snake*.

Take the intersection of the triangles that the primitive vector passes through to find the *Markov snake*.

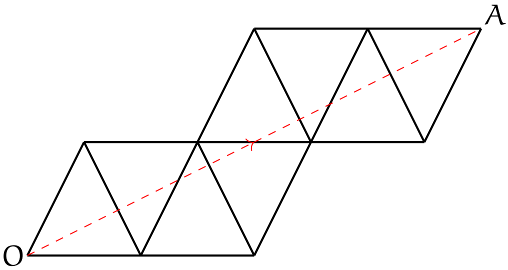


Figure: Markov snake for vector OA.

Combinatorial Interpretation of Markov

Form a bipartite graph by:

- Labelling vertices of the triangles with black nodes, but removing the two end vertices.
- Labelling the centres of the triangles with white nodes.
- Adding edges between a nodes if the black node corresponds to a vertex of the white nodes' triangle.

Form a bipartite graph by:

- Labelling vertices of the triangles with black nodes, but removing the two end vertices.
- Labelling the centres of the triangles with white nodes.
- Adding edges between a nodes if the black node corresponds to a vertex of the white nodes' triangle.

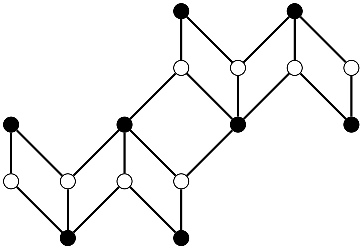


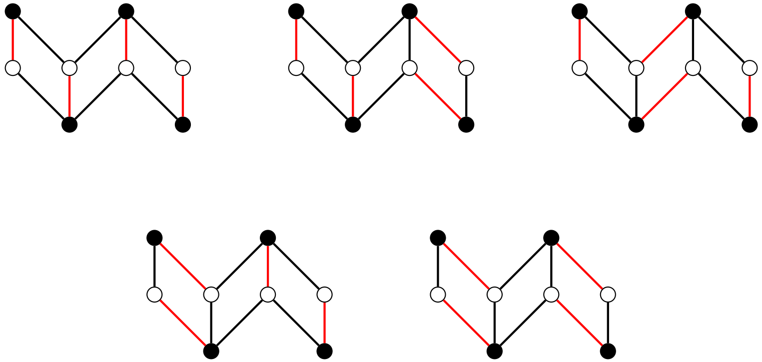
Figure: Bipartite graph from the Markov snake.

Combinatorial Interpretation of Markov

It can then be shown that the corresponding Markov number is equal to the number of perfect matchings of this bipartite graph.

It can then be shown that the corresponding Markov number is equal to the number of perfect matchings of this bipartite graph.

Here we shown this in the simpler case of $\rho = \frac{1}{2}$, in which case $m_\rho = 5$. The perfect matchings are shown by the red edges.



Combinatorial Interpretation of Markov

This combinatorial representation can be generalised to look at Markov polynomials.

This combinatorial representation can be generalised to look at Markov polynomials.

To do so we label the edges, based on their orientation as follows:

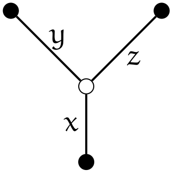


Figure: Weights assigned to edges in the bipartite graph.

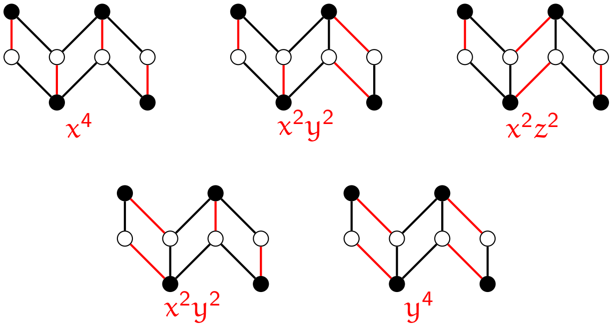
Combinatorial Interpretation of Markov

Each perfect matching then produces a monomial by multiplying together the weights of the edges involved in the matching.

Combinatorial Interpretation of Markov

Each perfect matching then produces a monomial by multiplying together the weights of the edges involved in the matching.

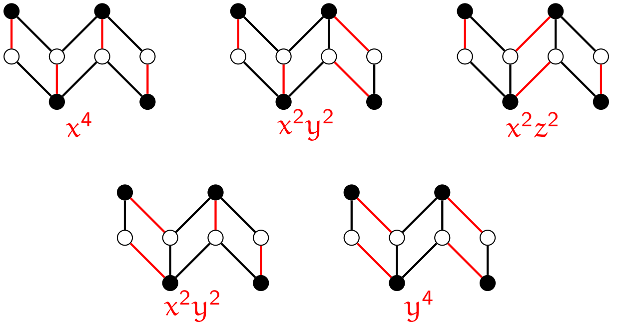
Returning to the case of $\rho = \frac{1}{2}$ we find



Combinatorial Interpretation of Markov

Each perfect matching then produces a monomial by multiplying together the weights of the edges involved in the matching.

Returning to the case of $\rho = \frac{1}{2}$ we find



Summing these together then gives the corresponding Markov polynomial

$$P_\rho = x^4 + 2x^2y^2 + y^4 + x^2z^2$$

References

- A. Itsara, G. Musiker, J. Propp, and R. Viana: *Combinatorial interpretations for the Markoff numbers.* (2003).
- J. Propp: *The combinatorics of frieze patterns and Markoff numbers* ArXiv:math/0511633 (2005).
- P. Caldero, A. Zelevinsky: *Laurent expansions in cluster algebras via quiver representations.* ArXiv:math/0604054 (2006).
- O.N. Karpenkov: *Geometry of Continued Fractions* (2013).
- S.J. Evans, A.P. Veselov, B. Winn: *Arithmetic and Geometry of Markov Polynomials* (in preparation).